Prova 2 - Versão 2

Probabilidade | Primeiro Semestre

Wyara Vanesa Moura e Silva | 2022

Questão 1

Seja $\mathbf{X} \sim \text{Exp}(\theta)$; $\theta > 0$. Definimos $\mathbf{T} = \lfloor \mathbf{X} \rfloor$; onde $a \in \mathbb{R}$, $\lfloor \mathbf{a} \rfloor = k \Leftrightarrow k \leq \mathbf{a} < k+1$. Calcular

$$\mathbb{P}(\mathbf{T} = k); \qquad \mathbb{E}(\mathbf{T})$$

Solução:

$$\begin{aligned}
\mathbb{P}(\mathbf{T} = k) &= \mathbb{P}(\lfloor \mathbf{X} \rfloor = k) &&= \mathbb{P}(k \leq \mathbf{X} < k+1) \\
&= \int_k^{k+1} \theta \cdot e^{-\theta x} dx &&= \theta \left(-\frac{1}{\theta} \cdot e^{-\theta x} \Big|_k^{k+1} \right) \\
&= -e^{-\theta(k+1)} + e^{-\theta k} &&= e^{-\theta k}(1 - e^{-\theta})
\end{aligned}$$

em que tal função corresponde a função de probabilidade da distribuição geométrica $(1 - e^{-\theta})$.

$$\begin{aligned}
\mathbb{E}(\mathbf{T}) &= \sum_{k=0}^{\infty} t \cdot e^{-\theta t}(1 - e^{-\theta}) &&= (1 - e^{-\theta}) \cdot \sum_{k=0}^{\infty} t \cdot e^{-\theta t} \\
&= (1 - e^{-\theta}) \cdot \left[\frac{d}{d\theta} \left(-\sum_{k=0}^{\infty} e^{-\theta t} \right) \right] &&= (1 - e^{-\theta}) \frac{e^{-\theta}}{(1 - e^{-\theta})^2} \\
&= \frac{e^{-\theta}}{(1 - e^{-\theta})}
\end{aligned}$$

Questão 2

Sejam $\mathbf{X}$ e $\mathbf{Y}$ i.i.d. geométricas $(\mathbf{p})$; $0 < \mathbf{p} < 1$.

$$\mathbb{P}(\mathbf{X} = k) = \mathbf{p}(1 - \mathbf{p})^{k-1}; \quad k = 1, 2, 3, \ldots$$

Sejam $\mathbf{Z} = \mathbf{Y} - \mathbf{X}$ e $\mathbf{W} = \min\{\mathbf{X}, \mathbf{Y}\}$. Encontrar a probabilidade conjunta $\mathbb{P}(\mathbf{W} = j, \mathbf{Z} = k)$.

Solução:

$$\begin{aligned}
\mathbb{P}(\mathbf{W} = j, \mathbf{Z} = k) &= \mathbb{P}(\min\{\mathbf{X}, \mathbf{Y}\} = j, \mathbf{Y} - \mathbf{X} = k, \mathbf{X} \leq \mathbf{Y}) + \\
&\quad + \mathbb{P}(\min\{\mathbf{X}, \mathbf{Y}\} = j, \mathbf{Y} - \mathbf{X} = k, \mathbf{X} > \mathbf{Y}) \\
&= \mathbb{P}(\mathbf{X} = j, \mathbf{Y} = k + \mathbf{X}, \mathbf{X} \leq \mathbf{Y})) + \mathbb{P}(\mathbf{Y} = j, \mathbf{X} = \mathbf{Y} - k, \mathbf{X} > \mathbf{Y})) \\
&= \mathbb{P}(\mathbf{X} = j, \mathbf{Y} = k + j, j \leq k + j) + \mathbb{P}(\mathbf{Y} = j, \mathbf{X} = j - k, j - k > j)) \\
&= \mathbb{P}(\mathbf{X} = j, \mathbf{Y} = k + j, 0 \leq k) + \mathbb{P}(\mathbf{Y} = j, \mathbf{X} = j - k, k < 0))
\end{aligned}$$

$$
\begin{aligned}
&= \mathbb{P}(\mathbf{X}=j)\cdot\mathbb{P}(\mathbf{Y}=k+j)\mathbb{1}_{\{0,1,2,\ldots\}}(k) \ + \ \mathbb{P}(\mathbf{Y}=j)\cdot\mathbb{P}(\mathbf{X}=j-k)\mathbb{1}_{\{-1,-2,\ldots\}}(k)\\
&= \mathbf{p}(1-\mathbf{p})^{j-1}\mathbf{p}(1-\mathbf{p})^{k+j-1}\cdot\mathbb{1}_{\{0,1,2,\ldots\}}(k) \ + \ \mathbf{p}(1-\mathbf{p})^{j-k-1}\mathbf{p}(1-\mathbf{p})^{j-1}\mathbb{1}_{\{-1,-2,\ldots\}}(k)\\
&= \mathbf{p}^2(1-\mathbf{p})^{k+2j-2}\mathbf{p}(1-\mathbf{p})^{k+j-1}\cdot\mathbb{1}_{\{0,1,2,\ldots\}}(k)+\\
&\quad+\mathbf{p}^2(1-\mathbf{p})^{-k+2j-2}\mathbf{p}(1-\mathbf{p})^{j-1}\mathbb{1}_{\{-1,-2,\ldots\}}(k)\\
&= \mathbf{p}^2(1-\mathbf{p})^{2j-2}\left[(1-\mathbf{p})^k\mathbb{1}_{\{0,1,2,\ldots\}}(k)+(1-\mathbf{p})^{-k}\mathbb{1}_{\{-1,-2,\ldots\}}(k)\right]
\end{aligned}
$$

Questão 3

$\mathbf{X}\sim$ Uniforme$[0,a]$; $a>0$, $\mathbf{Y}\sim\text{Exp}(\theta)$; $\theta>0$, independentes. Seja $\mathbf{Z}=\mathbf{X}+\mathbf{Y}$. Calcular a densidade de $\mathbf{X}$.

Solução:

$$
f_z(z) \quad = \quad \int_0^\infty f_{\mathbf{Y}}(y)\cdot f_{\mathbf{X}}(z-y)dy
$$

Logo,

se $0<z<a$

$$
f_z(z) \quad = \quad \int_0^z \frac{1}{a}\cdot\theta\cdot e^{-\theta x}dy = \frac{1}{a}\left[\theta\left(-\frac{1}{\theta}\cdot e^{-\theta x}\Big|_0^z\right)\right] \quad = \quad \frac{1}{a}\left(1-e^{-\theta z}\right)\mathbb{1}_{(0,a)}(z)
$$

se $a<z<\infty$

$$
\begin{aligned}
f_z(z) \quad &= \quad \int_{z-a}^z \frac{1}{a}\cdot\theta\cdot e^{-\theta x}dy = \frac{1}{a}\left[\theta\left(-\frac{1}{\theta}\cdot e^{-\theta x}\Big|_0^z\right)\right]\\
&= \quad \frac{1}{a}\cdot\left(e^{-\theta(z-a)}\right)\\
&= \quad \frac{1}{a}^{-\theta z}\left(e^{\theta a}-1\right)\mathbb{1}_{(a,\infty)}(z)
\end{aligned}
$$

Questão 4

Dada a densidade conjunta de $\mathbf{X},\mathbf{Y}$.

$$
f_{\mathbf{X},\mathbf{Y}}(x,y)=\frac{\sqrt{3}}{4\pi}\exp\left[-\frac{1}{2}(x^2-xy+y^2)\right]; \qquad x,y\in\mathbb{R}
$$

Calcular $\mathbb{E}[\mathbf{XY}]$.

Solução:

$$
\begin{aligned}
\mathbb{E}[\mathbf{XY}] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot f_{\mathbf{XY}}(xy)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \cdot \frac{\sqrt{3}}{4\pi}\exp\left[-\frac{1}{2}(x^2 - xy + y^2)\right] dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \cdot \frac{\sqrt{3}}{4\pi} \cdot x \cdot \exp\left[-\frac{1}{2}(x^2 - xy + y^2)\right] dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \cdot \frac{\sqrt{3}}{4\pi} \cdot x \cdot \exp\left[-\frac{1}{2}(x^2 - xy + y^2)\right] dxdy \\
&= \int_{-\infty}^{\infty} y \cdot \frac{\sqrt{3}}{4\pi} \cdot \int_{-\infty}^{\infty} x \cdot \exp\left[-\frac{1}{2}\left(x - \frac{y}{2}\right)^2 - \frac{1}{2}\left(\frac{3y^2}{4}\right)\right] dxdy \\
&= \frac{\sqrt{3}}{4\pi} \cdot \int_{-\infty}^{\infty} y \cdot \exp\left[-\frac{1}{2}\left(\frac{3y^2}{4}\right)\right] \sqrt{2\pi} \int_{-\infty}^{\infty} x \cdot \frac{1}{\sqrt{2\pi}}\exp\left[-\frac{1}{2}\left(x - \frac{y}{2}\right)^2\right] dxdy \\
&= \frac{\sqrt{3}\sqrt{2\pi}}{4\pi} \cdot \int_{-\infty}^{\infty} y \cdot \exp\left[-\frac{1}{2}\left(\frac{3y^2}{4}\right)\right] \frac{y}{2} dy \\
&= \frac{\sqrt{3}\sqrt{2\pi}}{4\pi} \cdot \int_{-\infty}^{\infty} \frac{y^2}{2} \cdot \exp\left[-\frac{1}{2}\left(\frac{y^2}{\frac{4}{3}}\right)\right] dy \\
&= \frac{\sqrt{3}\sqrt{2\pi}}{8\pi} \cdot \int_{-\infty}^{\infty} y^2 \cdot \frac{\sqrt{2\pi}\sqrt{\frac{4}{3}}}{\sqrt{2\pi}\sqrt{\frac{4}{3}}} \cdot \exp\left[-\frac{1}{2}\left(\frac{y^2}{\frac{4}{3}}\right)\right] dy \\
&= \frac{\sqrt{3}\sqrt{2\pi}}{8\pi} \cdot \sqrt{2\pi} \cdot \sqrt{\frac{4}{3}} \;=\; \frac{2}{3}
\end{aligned}
$$